Lecture Notes, Lectures 6, 7

2.1 Set Theory

Logical Inference

Let A and B be two logical conditions, like A="it's sunny today" and B="the light outside is very bright"

$\mathbf{A} \Rightarrow B$

A implies B, if A then B

$A \Leftrightarrow B$

A if and only if B, A implies B and B implies A, A and B are equivalent conditions

Definition of a Set

$\{ \}$

$\{x \mid x \text{ has property P}\}$

$\{1, 2, ..., 9, 10\} = \{ x \mid x \text{ is an integer}, 1 \leq x \leq 10 \}$.

Elements of a set

$x \in A$; $y \notin A$

$x \neq \{ x \}$

$x \in \{ x \}$

$\phi \equiv$ the empty set ($\equiv$ null set), the set with no elements.

Subsets

$A \subset B$ or $A \subseteq B$ if $x \in A \Rightarrow x \in B$

$A \subset A$ and $\phi \subset A$.

Set Equality

A = B if A and B have precisely the same elements

A = B if and only if $A \subset B$ and $B \subset A$.

Set Union

A $\cup$ B

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ ('or' includes 'and')

Set Intersection

$\cap$

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

If $A \cap B = \phi$ we say that A and B are disjoint.

Theorem 1: Let A, B, C be sets,

a. $A \cap A = A,\ A \cup A = A$ (idempotency)

b. $A \cap B = B \cap A,\ A \cup B = B \cup A$ (commutativity)

c. $A \cap (B \cap C) = (A \cap B) \cap C$ (associativity)

$A \cup (B \cup C) = (A \cup B) \cup C$

d. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (distributivity)

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Complementation (set subtraction)

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$A \backslash B = \{x \mid x \in A,\ x \notin B\}$

Cartesian Product

<u>ordered pairs</u>

$A \text{ x } B = \{(x,\ y) \mid x \in A,\ y \in B\}$.

Note: If x ≠ y, then (x, y) ≠ (y, x) .

R = The set of real numbers
$\mathbf{R}^N$ = N-fold Cartesian product of R with itself.
$\mathbf{R}^N$ = R x R x R x ... x R, where the product is taken N times.
The order of elements in the ordered N-tuple (x, y, ...) is essential. If $x \neq y,\ (x, y, \ldots) \neq (y, x, \ldots)$.

2.4 $\mathbf{R}^N$, Real N-dimensional Euclidean space

Read Starr's *General Equilibrium Theory*, section 2.4.

R^2 = plane
R^3 = 3-dimensional space
R^N = N-dimensional Euclidean space

Definition of R:

R = the real line

$\pm\infty \notin$ R

+, - , $\times$, $\div$

closed interval : $[a, b] \equiv \{x | x \in R, a \leq x \leq b\}$.

R is *complete*. Nested intervals property: Let $x^\nu < y^\nu$ and $[x^{\nu+1}, y^{\nu+1}] \subseteq [x^\nu, y^\nu]$, $\nu = 1, 2, 3, \ldots$. Then there is $z \in R$ so that $z \in [x^\nu, y^\nu]$, for all ν .

R^N = N-fold Cartesian product of R.

$x \in R^N$, $x = (x_1, x_2, \ldots, x_N)$

x_i is the ith co-ordinate of x.

x = point (or *vector*) in R^N

Algebra of elements of R^N

$$x + y = (x_1 + y_1,\ x_2 + y_2,\ \ldots,\ x_N + y_N)$$

$\mathbf{0}$ = (0, 0, 0, ..., 0) , the origin in N-space

$x - y \equiv x + (-y)$ = $(x_1 - y_1, x_2 - y_2, \ldots, x_N - y_N)$

$t \in R,\ x \in R^N$, then $tx \equiv (tx_1,\ tx_2,\ \ldots,\ tx_N)$.

$x,\ y \in R^N,\ \ x \cdot y = \sum_{i=1}^{N} x_i y_i$. If p $\in R^N$ is a price vector and y $\in R^N$ is an economic action, then p · y = $\sum_{n=1}^{N} p_n y_n$ is the value of the action y at prices p.

Norm in R^N, the measure of distance

$$|x| \equiv \|x\| \equiv \sqrt{x \cdot x} \equiv \sqrt{\sum_{i=1}^{N} x_i^2} \quad .$$

Let $x,\ y \in R^N$. The distance between x and y is $\|x-y\|$.

$$| x - y | = \sqrt{\Sigma_i (x_i - y_i)^2} \quad .$$

$$\|x - y\| \geq 0 \text{ all } x,\ y \in R^N$$

$| x - y | = 0$ if and only if $x = y$.

Limits of Sequences

x^ν , $\nu = 1, 2, 3, \ldots$,

Example: $x^\nu = 1/\nu$. 1, 1/2, 1/3, 1/4, 1/5, ... $x^\nu \to 0$.

Formally, let $x^i \in R,\ i = 1,\ 2,\ \ldots$. Definition: We say $x^i \to x^0$ if for any $\varepsilon > 0$, there is $q(\varepsilon)$ so that for all $q' > q(\varepsilon),\ |x^{q'} - x^0| < \varepsilon$.

So in the example $x^\nu = 1/\nu$, $q(\varepsilon) = 1/\varepsilon$

Let $x^i \in R^N,\ i = 1,\ 2,\ \ldots$. We say that $x^i \to x^0$ if for each co-ordinate $n = 1,\ 2,\ \ldots,\ N,\ x_n^i \to x_n^0$.

Theorem 2.2: Let $x^i \in R^N,\ i = 1,\ 2,\ \ldots$. Then $x^i \to x^0$ if and only if for any ε there is $q(\varepsilon)$ such that for all $q' > q(\varepsilon),\ \|x^{q'} - x^0\| < \varepsilon$.

x^o is a *cluster point* of $S \subseteq \mathbf{R}^N$ if there is a sequence

$x^\nu \in R^N$ so that $x^\nu \to x^o$.

Open Sets

Let $X \subset R^N$; X is *open* if for every $x \in X$ there is an $\varepsilon > 0$ so that $\|x - y\| < \varepsilon$ implies $y \in X$.

Open interval in R: (a, b) = { x | x ∈ R, a < x < b}

ϕ and R^N are open.

Closed Sets

Example: Problem - Choose a point x in the closed interval [a, b] (where 0 < a < b) to maximize x^2.

Solution: x = b.

Problem - Choose a point x in the open interval (a, b) to maximize x^2. There is no solution in (a, b) since $b \notin (a, b)$.

A set is closed if it contains all of its cluster points.

Definition: Let $X \subset R^N$. X is said to be a **closed** set if for every sequence x^ν, $\nu = 1, 2, 3, \ldots$, satisfying,

(i) $x^\nu \in X$, and

(ii) $x^\nu \to x^0$,

it follows that $x^0 \in X$.

Examples: A closed interval in R, [a, b] is closed

A closed ball in R^N of radius r, centered at $c \in R^N$, $\{x \in R^N | \ |x-c| \leq r\}$ is a closed set.

A line in R^N is a closed set

<u>But</u> a set may be neither open nor closed (for example the sequence $\{1/\nu\}$, $\nu=1, 2, 3, 4, \ldots$ is not closed in R, since 0 is a limit point of the sequence but is not an element of the sequence; it is not open since it consists of isolated points).

Note: Closed and open are not antonyms among sets. ϕ and R^N are each both closed and open.

Let $X \subseteq R^N$. The closure of X is defined as

$$\bar{X} \equiv \{\, y \mid \text{there is } x^\nu \in X,\ \nu = 1, 2, 3, \ldots, \text{ so that } x^\nu \to y \,\}.$$

For example the closure of the sequence in R, $\{1/\nu \mid \nu=1, 2, 3, 4, \ldots\}$ is

$$\{0\}\cup\{1/\nu \mid \nu=1, 2, 3, 4, \ldots\}.$$

<u>Concept of Proof by contradiction</u>: Suppose we want to show that **A** $\Rightarrow$ B. Ordinarily, we'd like to prove this directly. But it may be easier to show that [not (**A** $\Rightarrow$ B)] is false. How? Show that [A & (not B)] leads to a contradiction. A: x = 1, B:x+3=4. Then [A & (not B)] leads to the conclusion that $1+3\neq4$ or equivalently $1\neq1$, a contradiction. Hence [A & (not B)] must fail so A$\Rightarrow$ B. (Yes, it does feel backwards, like your pocket is being picked, but it works).

Theorem 2.3: Let $X \subset R^N$. X is closed if $R^N \setminus X$ is open.

Proof: Suppose $R^N \setminus X$ is open. We must show that X is closed. If $X=R^N$ the result is trivially satisfied. For $X \neq R^N$, let $x^\nu \in X$, $x^\nu \to x^o$. We must show that $x^o \in X$ if $R^N \setminus X$ is open. Proof by contradiction. Suppose not. Then $x^o \in R^N \setminus X$. But $R^N \setminus X$ is open. Thus there is an ε neighborhood about x^o entirely contained in $R^N \setminus X$. But then for ν large, $x^\nu \in R^N \setminus X$, a contradiction. Therefore $x^o \in X$ and X is closed. QED

Theorem 2.4: 1. $X \subset \overline{X}$
2. $X = \overline{X}$ if and only if X is closed.

Bounded Sets

Def: $K(k) = \{x | x \in R^N, \ |x_i| \leq k, \ i = 1, 2, \ldots, N\} =$ cube of side 2k (centered at the origin).

Def: $X \subset R^N$. X is *bounded* if there is $k \in R$ so that $X \subset K(k)$.

Compact Sets

THE IDEA OF COMPACTNESS IS ESSENTIAL!

Def: $X \subset R^N$. X is *compact* if X is closed and bounded.

Finite subcover property: An open covering of X is a collection of open sets so that X is contained in the union of the collection. It is a property of compact X that for every open covering there is a finite subset of the open covering whose union also contains X. That is, every open covering of a compact set has a finite subcover.

Boundary, Interior, etc.

$X \subset R^N$, Interior of X = $\{y | y \in X$, there is ε > 0 so that $\|x - y\| < \varepsilon$ implies $x \in X\}$

Boundary $X \equiv \overline{X}\backslash$Interior X

Set Summation in R^N

Let $A \subseteq R^N$, $B \subseteq R^N$. Then

$$A + B \equiv \{ x \mid x = a + b, a \in A, b \in B \}.$$

The Bolzano-Weierstrass Theorem, Completeness of R^N.

Theorem 2.5 (Nested Intervals Theorem): By an interval in R^N, we mean a set I of the form
$I = \{(x_1,\ x_2,\ \ldots,\ x_N) \mid a_1 \le x_1 \le b_1,$
$a_2 \le x_2 \le b_2,\ \ldots,\ a_N \le x_N \le b_N,\ a_i,\ b_i \in R\}$.
Consider a sequence of nonempty closed intervals I_k such that

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots \supseteq I_k \supseteq \ldots .$$

Then there is a point in R^N contained in all the intervals. That is, $\exists x^o \in \bigcap_{i=1}^{\infty} I_i$ and therefore $\bigcap_{i=1}^{\infty} I_i \neq \phi$; the intersection is nonempty.

Proof: Follows from the completeness of the reals, the nested intervals property on R.

Corollary (Bolzano-Weierstrass theorem for sequences): Let x^i , i = 1, 2, 3, ... be a bounded sequence in R^N . Then x^i contains a convergent subsequence.

Proof 2 cases: x^i assumes a finite number of values, x^i assumes an infinite number of values.

It follows from the Bolzano-Weierstrass Theorem for sequences and the definition of compactness that an infinite sequence on a compact set has a convergent subsequence whose limit is in the compact set.

2.3 Functions

We describe a function f(•) as follows:

For each $x \in A$ there is $y \in B$ so that $y = f(x)$.

$f: A \to B$.

A = domain of f

B = range of f

graph of f = S ⊂ $A x B$, S = {(x, y) | y = f(x)}

Let T⊂A.

f(T) ≡ {y | y = f(x), x ∈T} is the image of T under f.

f^{-1} : B → A, f^{-1} is known as "f inverse"

$f^{-1}(y) = \{x \mid x \in A,\ y = f(x)\}$

2.5 Continuous Functions

Let $f : A \to B,\ A \subset R^m,\ B \subset R^p$.

The notion of continuity of a function is that there are no jumps in the function values. Small changes in the argument of the function (x) result in small changes in the value of the function (y=f(x)).

Let ε, δ(ε), be small positive real numbers; we use the functional notation δ(ε) to emphasize that the choice of δ depends on the value of ε. f is said to be **continuous** at a ∈ A if

(i) for every $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that $|x - a| < \delta(\varepsilon) \Rightarrow |f(x) - f(a)| < \varepsilon$, or equivalently,

(ii) $x^\nu \in A,\ \nu = 1,\ 2,\ \ldots,$ and $x^\nu \to a,$ implies $f(x^\nu) \to f(a)$

.

Theorem 2.6: Let $f : A \rightarrow B$, f continuous. Let $S \subset B$, S closed. Then $f^{-1}(S)$ is closed.

Proof: Let $x^{\nu} \in f^{-1}(S)$, $x^{\nu} \rightarrow x^{o}$. We must show that $x^{o} \in f^{-1}(S)$. Continuity of f implies that $f(x^{\nu}) \rightarrow f(x^{o})$. $f(x^{\nu}) \in S$, S closed, implies $f(x^{o}) \in S$. Thus $x^{o} \in f^{-1}(S)$. QED

Note that as a consequence of Thm 2.6, the inverse image under a continuous function of an open subset of the range is open (since the complement of a closed set is open).

Theorem 2.7: Let $f: A \rightarrow B$, f continuous. Let $S \subset A$, S compact. Then f(S) is compact.
Proof: We must show that f(S) is closed and bounded.

Closed: Let $y^\nu \in f(S)$, $\nu=1,2,...$, $y^\nu \rightarrow y^o$. Show that $y^o \in f(S)$. There is $x^\nu \in S$, $x^\nu = f^{-1}(y^\nu)$. Take a convergent subsequence, relabel, and $x^\nu \rightarrow x^o \in S$ by closedness of S. But continuity of f implies that $f(x^\nu) \rightarrow f(x^o) = y^o \in f(S)$.

Bounded: For each $y \in f(S)$, let $C(y)=\{z \mid z \in B, |y-z|<\varepsilon\}$, an ε-ball about y. The family of sets $\{f^{-1}(C(y)) \mid y \in f(S)\}$ is an open cover of S (the inverse image of an open set under f is open since the inverse image of its complement --- a closed set --- is closed, Thm 2.6). There is a finite subcover. Hence f(S) is covered by a finite family of ε balls. f(S) is bounded. QED

Corollary 2.2: Let $f: A \rightarrow R$, f continuous, $S \subset A$, S compact, then there are $\bar{x}, \underline{x} \in S$ such that $f(\bar{x}) = \sup\{f(x) | x \in S\}$ and $f(\underline{x}) = \inf\{f(x) | x \in S\}$, where inf indicates greatest lower bound and sup indicates least upper bound.

Corollary 2.2 is very important for economic analysis. It provides sufficient conditions so that maximizing behavior takes on well defined values.

Homogeneous Functions

f: $R^p \rightarrow R^q$.
f is homogeneous of degree 0 if for every scalar (real number) $\lambda > 0$, we have $f(\lambda x) = f(x)$.
f is homogeneous of degree 1 if for every scalar $\lambda > 0$, we have $f(\lambda x) = \lambda f(x)$.